Real zeros of $_2F_1$ hypergeometric polynomials

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Abstract

We use a method based on the division algorithm to determine all the values of the real parameters b and c for which the hypergeometric polynomials $_2F_1(-n, b; c; z)$ have n real, simple zeros. Furthermore, we use the quasi-orthogonality of Jacobi polynomials to determine the intervals on the real line where the zeros are located.

Keywords: Orthogonal polynomials, zeros, hypergeometric polynomials.
2000 MSC: 33C05, 33C45, 42C05.

1. Introduction

The $_2F_1$ hypergeometric function is defined by (cf. [1])

$$_2F_1\,(a, b; c; z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1,$$

where a, b and c are complex parameters, $-c \notin \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and

$$(\alpha)_k = \begin{cases} \alpha(\alpha + 1) \ldots (\alpha + k - 1) & , \quad k \in \mathbb{N}, \\ 1 & , \quad k = 0, \ \alpha \neq 0 \end{cases}$$

is Pochhammer's symbol. This series converges when $|z| < 1$ and also when $z = 1$ provided that $\mathrm{Re}(c - a - b) > 0$ and when $z = -1$ provided that $\mathrm{Re}(c - a - b + 1) > 0$. When one of the numerator parameters is equal to a nonpositive integer, say $a = -n$, $n \in \mathbb{N}_0$, the series terminates and the function is a polynomial of degree n in z.

The problem of describing the zeros of the polynomials $_2F_1\,(-n, b; c; z)$ when b and c are complex arbitrary parameters, has not been solved. Even when b and c are both real, the only cases that have been fully analyzed impose additional restrictions on b and c. Recent publications (cf. [4], [6], [7], [8], [11] and [13]) considered the zero location of special classes of $_2F_1\,(-n, b; c; z)$ with restrictions on the parameters b and c. Results on the asymptotic zero distribution of certain classes of $_2F_1\,(-n, b; c; z)$ have also appeared (cf. [5], [10], [14], [15] and [27]).

Different types of $_2F_1(-n, b; c; z)$ have well-established connections with classical orthogonal polynomials, notably the Jacobi polynomials and the Gegenbauer or ultraspherical polynomials (cf. [1]). For the ranges of the parameters where these polynomials are orthogonal, information about the zeros of $_2F_1(-n, b; c; z)$ follows immediately from classical results (cf. [1], [28]). The asymptotic zero distribution of $_2F_1(-n, b; c; z)$ when b and c depend on n can be deduced from recent results by Kuijlaars, Martínez-Finkelshtein, Martínez-González and Orive (cf. [20], [21], [22], [23]) on the asymptotic zero distribution of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ when the parameters α and β depend on n. Conversely, if the distribution of the zeros of $_2F_1(-n, b; c; z)$ is known, this leads to information about the zero distribution of other special functions (cf. [6]). This makes knowledge of the zero distribution of $_2F_1(-n, b; c; z)$ extremely valuable.

The orthogonality of the polynomials $_2F_1(-n, b; c; z)$ given in the next theorem follows from the orthogonality of the Jacobi polynomials (cf. [25, p. 257-261]) and can also be proved directly using the Rodrigues' formula for the polynomials $_2F_1(-n, b; c; z)$ (cf. [1, p. 99]) as was done in [9] and [21].

Theorem 1 (cf. [9]). *Let $n \in \mathbb{N}_0$, b, $c \in \mathbb{R}$ and $-c \notin \mathbb{N}_0$. Then $_2F_1(-n, b; c; z)$ is the nth degree orthogonal polynomial for the n-dependent positive weight function $|z^{c-1}(1-z)^{b-c-n}|$ on the intervals*

  (i) $(-\infty, 0)$ for $c > 0$ and $b < 1 - n$;
  (ii) $(0, 1)$ for $c > 0$ and $b > c + n - 1$;
  (iii) $(1, \infty)$ for $c + n - 1 < b < 1 - n$.

As a consequence of orthogonality, we know that for each n, the n zeros of $_2F_1(-n, b; c; z)$ are real, simple and lie in the interval of orthogonality for the corresponding ranges of the parameters (see, for example, [12], Theorem 4) as illustrated in Figure 1.
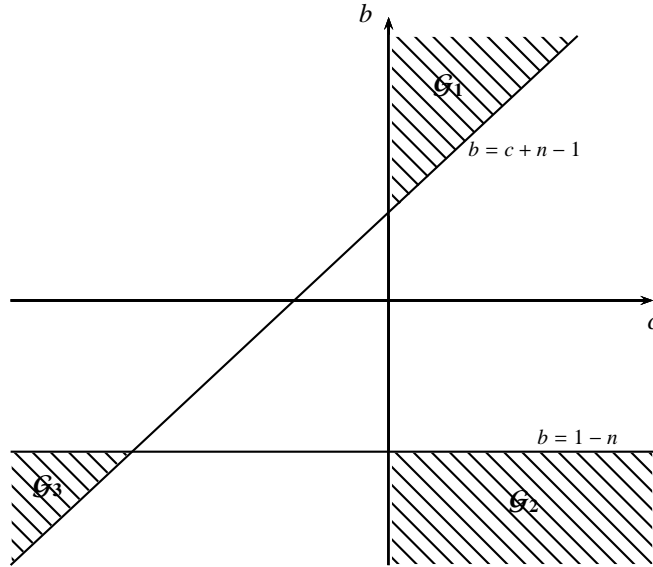

Figure 1: Values of b and c for which $_2F_1(-n, b; c; z)$ is orthogonal and has n real simple zeros in the intervals $(0, 1)$, $(-\infty, 0)$ and $(1, \infty)$ are indicated by regions $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ respectively .

In his classical paper (cf. [19]), Felix Klein obtained results on the precise number of zeros of $_2F_1(a, b; c; z)$ that lie in each of the intervals $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$ by generalizing earlier results of Hilbert (cf. [17]). These Hilbert-Klein formulas are valid for hypergeometric functions and not only for polynomials. Szegö recaptured these results for the special case of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, which have a representation as $_2F_1(-n, b; c; z)$, in the intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$ (cf. [28], p.145, Theorem 6.72). The number and location of the real zeros of $_2F_1(-n, b; c; z)$ for b and c real can be deduced as follows.

Theorem 2 (cf. [11], Theorem 3.2). *Let $n \in \mathbb{N}$, b, $c \in \mathbb{R}$ and $c > 0$. Then,*

(i) *For $b > c + n$, all zeros of $_2F_1(-n, b; c; z)$ are real and lie in the interval $(0, 1)$.*

(ii) *For $c < b < c + n$, $c + j - 1 < b < c + j$, $j = 1, 2, \ldots, n$, $_2F_1(-n, b; c; z)$ has j real zeros in $(0, 1)$. The remaining $(n - j)$ zeros of $_2F_1(-n, b; c; z)$ are all non-real if $(n - j)$ is even, while if $(n - j)$ is odd, $_2F_1(-n, b; c; z)$ has $(n - j - 1)$ non-real zeros and one additional real zero in $(1, \infty)$.*

(iii) *For $0 < b < c$, all the zeros of $_2F_1(-n, b; c; z)$ are non-real if n is even, while if n is odd, $_2F_1(-n, b; c; z)$ has one real zero in $(1, \infty)$ and the other $(n - 1)$ zeros are non-real.*

(iv) *For $-n < b < 0$, $-j < b < -j + 1$, $j = 1, 2, \ldots, n$, $_2F_1(-n, b; c; z)$ has j real negative zeros. The remaining $(n - j)$ zeros of $_2F_1(-n, b; c; z)$ are all non-real if $(n - j)$ is even, while if $(n - j)$ is odd, $_2F_1(-n, b; c; z)$ has $(n - j - 1)$ non-real zeros and one additional real zero in $(1, \infty)$.*

(v) *For $b < -n$, all zeros of $_2F_1(-n, b; c; z)$ are real and negative.*

The values of the parameters b and c for which $_2F_1(-n, b; c; z)$ has exactly n real simple zeros in $(0, 1)$ given in Theorem 2 (i) and (ii) correspond to those in Theorem 1(ii) while the parameter values in Theorem 1(i) that ensure that all the zeros of $_2F_1(-n, b; c; z)$ are real, simple and negative are the same as those in Theorem 2 (iv) and (v). The values of b and c in Theorem 1 (iii) for which n zeros are in $(1, \infty)$ can also be obtained from Theorem 2 (iv) and (v) using the transformation (cf. [1, p. 79, (2.3.14)])

$$_2F_1(-n, b; c; z) = \frac{(c - b)_n}{(c)_n} {}_2F_1(-n, b; 1 - n + b - c; 1 - z)$$

due to Pfaff.

A natural question to ask is whether the parameter ranges in Theorems 1 and 2 are the only values of b, $c \in \mathbb{R}$ for which $_2F_1(-n, b; c; z)$ have n real simple zeros. In this paper, we use a method that does not rely on orthogonality to determine all the real values of the parameters b and c for which $_2F_1(-n, b; c; z)$ have n real simple zeros. We apply an algorithm which counts the zeros of polynomials with real coefficients and their multiplicities. We also determine the intervals where the real zeros are located for these values of b and c.

2. The algorithm

Recall that given two polynomials $f(x)$ and $g(x)$, with $\deg(f) \geq \deg(g)$, there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = q(x)g(x) + r(x)$ with $\deg(r) < \deg(g)$. We will denote the leading coefficient of a polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ by $lc(f) = a_n$.

We use the following algorithm (cf. [24]).

Let $f(x)$ be a real polynomial with $\deg(f) = n \geq 2$. Define

$$f_0(x) := f(x) \qquad \text{and} \qquad f_1(x) := f'(x)$$

and proceed for $k \in \mathbb{N}$ as follows:

If $\deg(f_k) > 0$ perform the division of f_{k-1} by f_k to obtain

$$f_{k-1}(x) = q_{k-1}(x)f_k(x) - r_k(x).$$

Define

$$f_{k+1}(x) = \begin{cases} r_k(x) & \text{if } r_k(x) \not\equiv 0 \\ f_k'(x) & \text{if } r_k(x) \equiv 0 \end{cases}$$

and generate the sequence of numbers $c_1, c_2, \ldots$ where

$$c_k = \begin{cases} \dfrac{lc(f_{k+1})}{lc(f_{k-1})} & \text{if } r_k(x) \not\equiv 0 \\ 0 & \text{if } r_k(x) \equiv 0 \end{cases}.$$

When f_k is constant, the algorithm terminates.

Note that the algorithm must terminate, since the degrees of the polynomials $f_k(x)$ decrease on each step.

Then we have the following theorem which we will apply to $_2F_1(-n, b; c; z)$.

Theorem 3 (cf. [24], Theorem 10.5.7, p.339). *Let f be a polynomial of degree n with real coefficients. Then f has only real zeros if and only if the above algorithm produced $n - 1$ non-negative numbers $c_1, \ldots, c_{n-1}$. Moreover, the zeros of f are all real and simple if and only if the numbers $c_1, \ldots, c_{n-1}$ are all positive.*

3. Main results

We shall assume throughout our discussion that $b, c \in \mathbb{R}$ with $b, c \neq 0, -1, \ldots, -n + 1$. The assumption on b is made to ensure that $_2F_1(-n, b; c; z)$ is a polynomial of degree n.

Proposition 4. *Let $b, c \in \mathbb{R}$. Then,*

1. *The zeros of $_2F_1(-2, b; c; z)$ are real and simple if and only if either (see Figure 2):*

    (i) $c < -1$ and $c < b < 0$.
    (ii) $-1 < c < 0$ and $b > 0$ or $b < c$.
    (iii) $c > 0$ and $b < 0$ or $c < b$.

2. *The zeros of $_2F_1(-3, b; c; z)$ are real and simple if and only if either (see Figure 3):*

    (i) $c < -2$ and $1 + c < b < -1$.
    (ii) $-2 < c < -1$ and $-1 < b < 1 + c$.
    (iii) $c > -1, c \neq 0$ and $b < -1$ or $b > c + 1$.
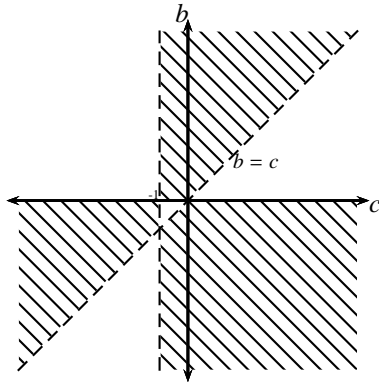

Figure 2: Values of b and c for which $_2F_1(-2, b; c; z)$ has only real simple zeros
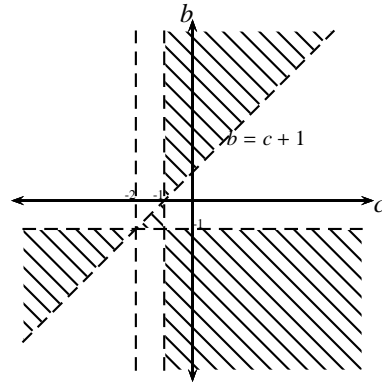
Figure 3: Values of b and c for which $_2F_1(-3, b; c; z)$ has only real simple zeros

Theorem 5. *For any integer $n \geq 4$, the polynomial $_2F_1(-n, b; c; z)$ has only real and simple zeros if and only if (c, b) belongs to one of the four n-dependent regions $\mathcal{R}_1, \ldots, \mathcal{R}_4$ defined by*

$$\mathcal{R}_1 = \{c + n - 2 < b < 2 - n\},$$
$$\mathcal{R}_2 = \{c > -1, \quad b < 2 - n\},$$
$$\mathcal{R}_3 = \{c > -1, \quad b > n - 2, \quad b > c + n - 2\},$$
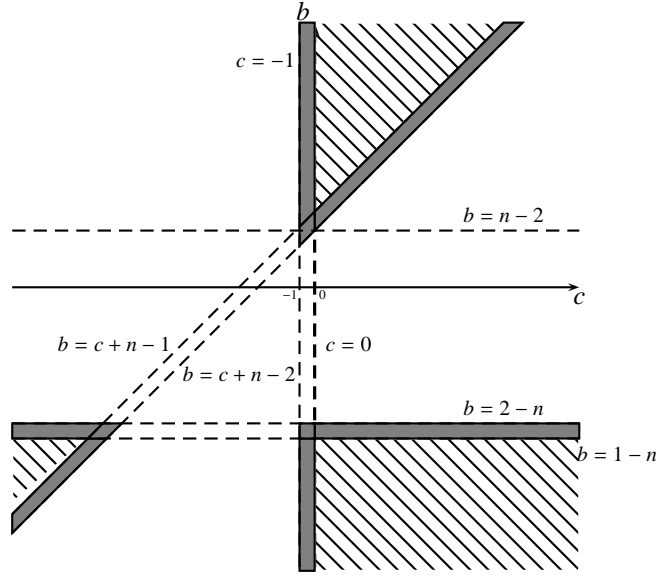$$\mathcal{R}_4 = \{-1 < c < 0, \quad c + n - 2 < b < n - 2\}.$$

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Figure 4: Values of b and c for which $_2F_1(-n, b; c; z)$, $n = 4, 5, \ldots$ has n real simple zeros

The parameter values $(c, b) \in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4$ described in Theorem 5 for which $_2F_1(-n, b; c; z)$, $n = 4, 5, \ldots$ has n real simple zeros are illustrated by the grey and diagonally shaded regions in Figure 4 with the grey regions indicating those parameter values that extend the results in Theorems 1 and 2.

Next, we turn our attention to the location of the zeros of $_2F_1(-n, b; c; z)$ for those parameter values located in the grey shaded regions in Figure 4 where the polynomials are no longer orthogonal and the location of the real zeros cannot be obtained using Theorems 1 and 2. For these values of b and c, the polynomials $_2F_1(-n, b; c; z)$ are quasi-orthogonal of order 1 and, in some cases, order 2 (cf. [3] and [2]). Theorem 3 in [2] and Theorem 6 in [18] yield information on the zero location of quasi-orthogonal polynomials with non-varying weight functions. However, these results cannot be applied to $_2F_1(-n, b; c; z)$ since their weight function depends on n. We use information about the zeros of Jacobi polynomials to obtain the following three results.

Theorem 6. *Let $n \in \mathbb{N}$ and $b, c \in \mathbb{R}$. Then, $_2F_1(-n, b; c; z)$ has all its zeros real and simple and $(n - 2)$ of them lie in*

(i) $(0, 1)$ *for $-1 < c < 0$ and $c + n - 2 < b < c + n - 1$. One of the remaining zeros lies in $(1, \infty)$ and the other one in $(-\infty, 0)$.*

(ii) $(1, \infty)$ *for $1 - n < b < 2 - n$ and $b - n + 1 < c < b - n + 2$. One of the remaining zeros lies in $(-\infty, 0)$ and the other one in $(0, 1)$.*

(iii) $(-\infty, 0)$ *for $-1 < c < 0$ and $1 - n < b < 2 - n$. One of the remaining zeros lies in $(1, \infty)$ and the other one in $(0, 1)$.*

Theorem 6 applies to the parameter values illustrated in Figure 5.

Theorem 7. *Let $n \in \mathbb{N}$ and $b, c \in \mathbb{R}$. Then, $_2F_1(-n, b; c; z)$ has all its zeros real and simple and $(n - 1)$ of them lie in*

(i) $(0, 1)$ *for $-1 < c < 0$ and $b > c + n - 1$. The remaining zero is negative.*

(ii) $(1, \infty)$ *for $1 - n < b < 2 - n$ and $c < b - n - 1$. The remaining zero is negative.*

(iii) $(-\infty, 0)$ *for $-1 < c < 0$ and $b < 1 - n$. The remaining zero lies in $(0, 1)$.*

The parameter values described in Theorem 7 are illustrated in Figure 6.

Theorem 8. *Let $n \in \mathbb{N}$ and $b, c \in \mathbb{R}$. Then, $_2F_1(-n, b; c; z)$ has all its zeros real and simple and $(n - 1)$ of them lie in*

(i) $(0, 1)$ *for $c > 0$ and $c + n - 2 < b < c + n - 1$. The remaining zero is in the interval $(1, \infty)$.*
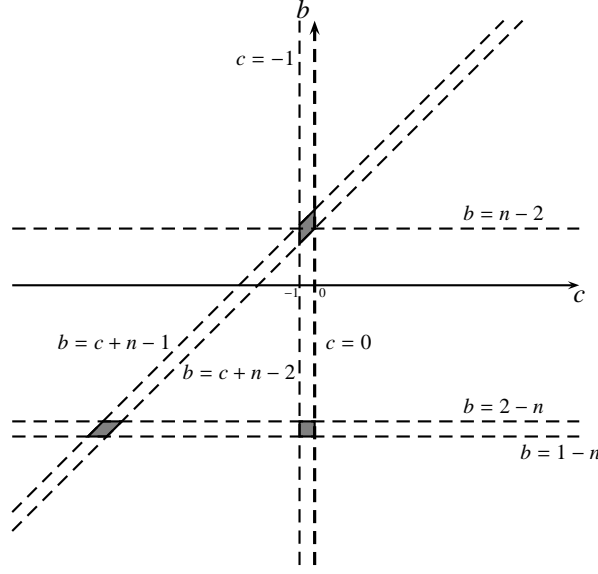
Figure 5: Values of b and c corresponding to those described in Theorem 6

(ii) $(1, \infty)$ for $b < 1 - n$ and $c + n - 2 < b < c + n - 1$. The remaining zero lies in $(0, 1)$.
(iii) $(-\infty, 0)$ for $c > 0$ and $1 - n < b < 2 - n$. The remaining zero lies in $(1, \infty)$.

Figure 7 illustrates the range of the parameters b and c referred to in Theorem 8.

4. Proofs

Proof of Proposition 4.

1. Since
$$_2F_1(-2, b; c; z) = 1 - \frac{2b}{c}z + \frac{b(b+1)}{c(c+1)}z^2,$$

   we see that $_2F_1(-2, b; c; z) = 0$ if and only if
   $$z = \frac{b(c+1) \pm \sqrt{b(c+1)(b-c)}}{b(b+1)}.$$

   Hence, the zeros of $_2F_1(-2, b; c; z)$ are real and simple if and only if $b(c+1)(b-c) > 0$.

2. The discriminant of
$$_2F_1(-3, b; c; z) = 1 - \frac{3b}{c}z + \frac{3b(b+1)}{c(c+1)}z^2 - \frac{b(b+1)(b+2)}{c(c+1)(c+2)}z^3$$

   is given by $\Delta_3 = 108 \dfrac{b^2(b+1)(b-c-1)(b-c)^2}{c^4(c+1)^3(c+2)^2}$ (cf. [16]) and therefore $_2F_1(-3, b; c; z)$ has real simple roots if and only if $\Delta_3 > 0$.

   ∎

The following two lemmas will be used in the proof of our main result.
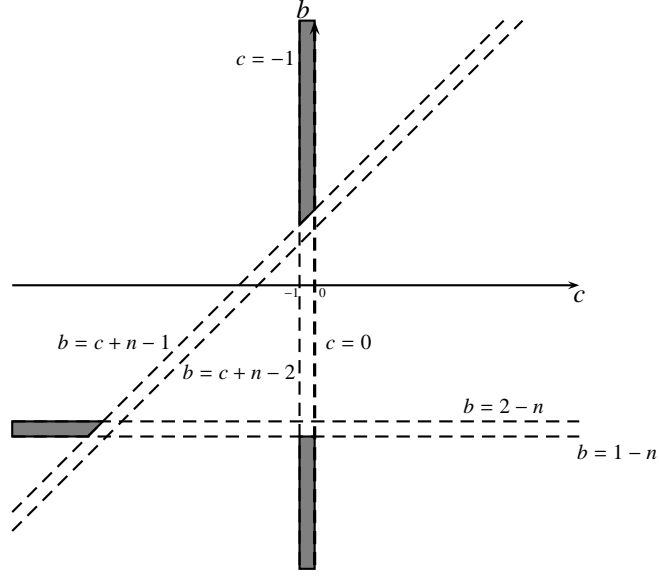
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Figure 6: Values of b and c corresponding to those described in Theorem 7

Lemma 9. *Let*

$$\alpha_{k,l} = \frac{\left(\frac{k-n}{2}\right)_l \left(\frac{2k-n-b-3}{4}\right)_l \left(\frac{k-b-1+c}{2}\right)_l}{\left(\frac{k-b-2}{2}\right)_l \left(\frac{2k-b-1-n}{4}\right)_l \left(\frac{k-n-1-c}{2}\right)_l}$$

and let the sequence θ_k be recursively defined by

$$\theta_{k+1} = \alpha_{k,1}\,\theta_{k-1}, \tag{1}$$

for $k \in \{2, \ldots, n-2\}$, with

$$\theta_1 = -\frac{nb}{c}, \quad \theta_2 = \frac{(b-c)(n-1)}{c(b+n-1)}.$$

Then,

$$\theta_{2k} = \frac{(b+1)(n+c)}{c(n+b+1)}\alpha_{1,k}, \quad k = 1, 2, \ldots \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{2}$$

and

$$\theta_{2k+1} = -\frac{nb}{c}\alpha_{2,k}, \quad k = 0, 1, \ldots \left\lfloor \frac{n-2}{2} \right\rfloor. \tag{3}$$

Proof of Lemma 9. We prove the result by induction on k. When $k = 1$, the right-hand side of (2) is

$$\frac{(b+1)(n+c)\left(\frac{1-n}{2}\right)\left(-\frac{n+b+1}{4}\right)\left(\frac{c-b}{2}\right)}{c(n+b+1)\left(-\frac{b+1}{2}\right)\left(-\frac{n+b-1}{4}\right)\left(-\frac{n+c}{2}\right)} = \frac{(b-c)(n-1)}{c(b+n-1)}$$

which is θ_2 as required.

We now assume the result is true for $k = t$ and prove the result true for $k = t+1$. If we let $k = t+1$ on the
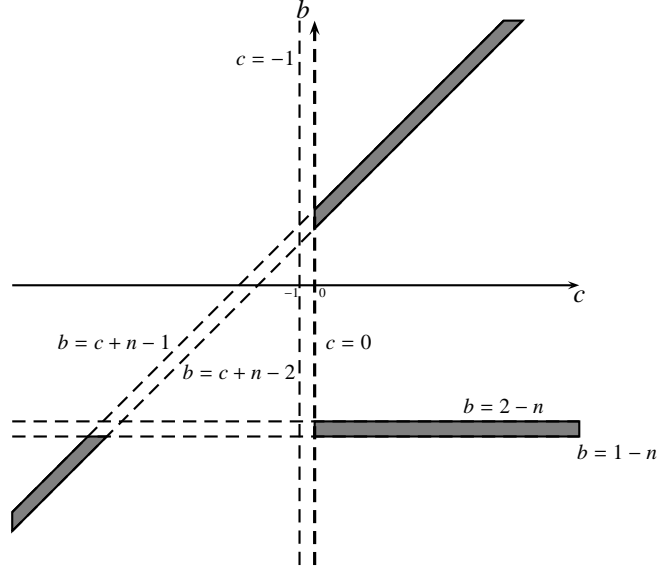
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Figure 7: Values of b and c corresponding to those described in Theorem 8

right-hand side of (2), we obtain

$$
\begin{aligned}
\text{RHS} &= \frac{(b+1)(n+c)}{c(n+b+1)}\,\alpha_{1,t+1} \\
&= \frac{(b+1)(n+c)}{c(n+b+1)}\,\frac{\left(\frac{1-n}{2}+t\right)\left(\frac{c-b}{2}+t\right)\left(-\frac{n+b+1}{4}+t\right)}{\left(-\frac{b+1}{2}+t\right)\left(-\frac{n+c}{2}+t\right)\left(-\frac{n+b-1}{4}+t\right)}\,\alpha_{1,t} \quad \text{since } (a)_{k+1} = (a+k)(a)_k \\
&= \theta_{2t}\,\frac{(1-n+2t)\,(c-b+2t)\,(-n-b-1+4t)}{(-b-1+2t)\,(-n-c+2t)\,(-n-b+1+4t)} \quad \text{by the inductive hypothesis} \\
&= \alpha_{2t+1,1}\,\theta_{2t} \\
&= \theta_{2t+2} \quad \text{from (1)}
\end{aligned}
$$

and the result follows by induction.

The second relation (3) may be proved by induction in a similar way. ∎

Lemma 10. *Let $n \geq 4$. Then, for all $k \in \{2, \ldots, n-1\}$,*

$$
\frac{(n-k)(n+c-k)(b+1-k)(b-c+1-k)}{(n+b+2-2k)(n+b-2k)(n+b+1-2k)^2} \tag{4}
$$

and

$$
\frac{(n-1)\,(n+c-1)\,(b-c)}{(n+b-2)\,(n+b-1)^2} \tag{5}
$$

are positive if and only if $(c,b) \in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4$.

Proof of Lemma 10. Since $n-1 > 0$ and $(n+b-1)^2 > 0$ for all $n \in \mathbb{N}$, $b \in \mathbb{R}$, we see that (5) is positive if and only

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if

$$(c, b) \in \{c < 1 - n, \ b > c, \ b < 2 - n\} = \mathcal{A}_1 \supset \mathcal{R}_1 \text{ or}$$
$$\in \{c > 1 - n, \ b < c, \ b < 2 - n\} = \mathcal{A}_2 \supset \mathcal{R}_2 \text{ or}$$
$$\in \{c > 1 - n, \ b > c, \ b > 2 - n\} = \mathcal{A}_3 \supset (\mathcal{R}_3 \cup \mathcal{R}_4) \text{ or}$$
$$\in \{c < 1 - n, \ b < c, \ b > 2 - n\} = \emptyset.$$

Clearly $\dfrac{(n - k)}{(n + b + 1 - 2k)^2} > 0$ for all $k \in \{2, \ldots, n - 1\}$, $n \in \mathbb{N}$ and $b \in \mathbb{R}$, $b \neq, 3 - n, 5 - n, \ldots, n - 5, n - 3$.

Furthermore $b > n - 2$ if and only if $n + b + 2 - 2k > 0$, $n + b - 2k > 0$ and $b + 1 - k > 0$ for all $k \in \{2, \ldots, n - 1\}$. Hence, when $b > n - 2$, (4) will be positive for all $k \in \{2, \ldots, n - 1\}$ if and only if

$$(c, b) \in \{b > c + k - 1, \ c + n - k > 0, \ k = 2, \ldots, n - 1\} = \{b > c + n - 2, \ c > -1\} = \mathcal{R}_3 \ \text{ or}$$
$$\in \{b < c + k - 1, \ c + n - k < 0, \ k = 2, \ldots, n - 1\} = \{b < c + 1, \ c < 2 - n\} = \emptyset \ \text{ or}$$
$$\in \{b < c + k - 1, \ c + n - k < 0, \ k = 2, \ldots, l\} \cap \{b > c + k - 1, \ c + n - k > 0, \ k = l + 1, \ldots, n - 1\} = \emptyset \ \text{ or}$$
$$\in \{b > c + k - 1, \ c + n - k > 0, \ k = 2, \ldots, l\} \cap$$
$$\{b < c + k - 1, \ c + n - k < 0, \ k = l + 1, \ldots, n - 1\} \cap \{b > n - 2\} = \emptyset.$$

Similarly, $b < 2 - n$ if and only if $b + 1 - k < 0$, $n + b + 2 - 2k < 0$ and $n + b - 2k < 0$ for $k \in \{2, \ldots, n - 1\}$. Hence, when $b < 2 - n$, (4) will be positive for all $k \in \{2, \ldots, n - 1\}$ if and only if

$$(c, b) \in \{b > c + k - 1, \ c + n - k < 0, \ k = 2, \ldots, n - 1\} = \{b > c + n - 2, \ c < 2 - n\} = \mathcal{A}_4 \supset \mathcal{R}_1 \ \text{ or}$$
$$\in \{b < c + k - 1, \ c + n - k > 0, \ k = 2, \ldots, n - 1\} = \{b < c + 1, \ c > -1\} = \mathcal{A}_5 \supset \mathcal{R}_2 \ \text{ or}$$
$$\in \{b > c + k - 1, \ c + n - k < 0, \ k = 2, \ldots, l\} \cap \{b < c + k - 1, \ c + n - k > 0, \ k = l + 1, \ldots, n - 1\} = \emptyset \ \text{ or}$$
$$\in \{b < c + k - 1, \ c + n - k > 0, \ k = 2, \ldots, l\} \cap \{b > c + k - 1, \ c + n - k < 0, \ k = l + 1, \ldots, n - 1\} = \emptyset.$$

For the remaining case where $2 - n < b < n - 2$ or, more specifically, $-2 < b + n - 2k < 0$ with $b - k + 1 < 0$ for all $k \in \{2, \ldots, n - 1\}$, the only non-empty possibility is that (4) is positive for $k = n - 1$ if and only if $c > -1$, $b > c + n - 2$ and $n - 4 < b < n - 2$ whereas (4) is positive for $k \in \{2, 3, \ldots, n - 2\}$ if and only if $c > -1$, $b > c + n - 2$ and $b > n - 3$. Hence, when $2 - n < b < n - 2$, (4) is positive for all $k \in \{2, \ldots, n - 1\}$ if and only if $(c, b) \in \mathcal{R}_4$.

Since $\mathcal{A}_1 \cap \mathcal{A}_4 = \mathcal{R}_1$ and $\mathcal{A}_2 \cap \mathcal{A}_5 = \mathcal{R}_2$, the result follows. ∎

Proof of Theorem 5. We apply the algorithm to the polynomial

$$f(z) = {}_2F_1(-n, b; c; z).$$

We have (cf. [25], p.69, ex.1)

$$f_1(z) = f'(z) = -\frac{nb}{c} \, {}_2F_1(-n + 1, b + 1; c + 1; z).$$

Using Raimundas Vidūnas' Maple package for contiguous relations of ${}_2F_1$ hypergeometric series (cf. [29], [30]), we obtain

$$f_0(z) = \frac{1}{n}\left(z - \frac{c + n - 1}{b + n - 1}\right) f_1(z) - \frac{(b - c)(n - 1)}{c(b + n - 1)} \, {}_2F_1(-n + 2, b; c + 1; z).$$

This relation can easily be verified by comparing coefficients. Thus,

$$f_2(z) = r_1(z) = \frac{(b - c)(n - 1)}{c(b + n - 1)} \, {}_2F_1(-n + 2, b; c + 1; z).$$

In the next step $(k = 2)$, we get

$$f_1(z) = q_1(z)f_2(z) - r_2(z),$$

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with

$$q_1(z) = \frac{n(b+n-1)^2(b+n-2)}{(n-1)(c+n-1)(b-c)}\left[z + \frac{(n-2)(c+n-2)}{b+n-3} - \frac{(n-1)(c+n-1)}{b+n-1}\right]$$

and

$$r_2(z) = \frac{(b+n-1)(b-1-c)n(n-2)}{c(c+n-1)(b+n-3)}\,{}_2F_1(-n+3, b-1; c+1; z).$$

Setting

$$f_k(z) = \theta_k\,{}_2F_1(-n+k, b+2-k; c+1; z), \quad k \in \{1, \ldots, n-1\},$$

we see that in general we need a contiguous relation of the form

$${}_2F_1(-n+k-1, b+3-k; c+1; z)$$
$$= q_{k-1}(z)\frac{\theta_k}{\theta_{k-1}}\,{}_2F_1(-n+k, b+2-k; c+1; z) - \frac{\theta_{k+1}}{\theta_{k-1}}\,{}_2F_1(-n+k+1, b+1-k; c+1; z),$$

for $k \in \{2, \ldots, n-2\}$, with

$$\theta_1 = -\frac{nb}{c}, \quad \theta_2 = \frac{(b-c)(n-1)}{c(b+n-1)}.$$

Using Vidũnas' package, we obtain (1) for $k = 2, 3, \ldots$ and from Lemma 9 we conclude that θ_k is well defined and non-zero for all $k \in \{1, \ldots, n-1\}$ when $c \neq 0$ and $(c, b) \in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4$. Thus,

$$c_k = \frac{lc(f_{k+1})}{lc(f_{k-1})}, \quad k \in \mathbb{N},$$

which implies that

$$c_1 = \frac{(b-c)(n-1)}{c(b+n-1)}\frac{(-n+2)_{n-2}(b)_{n-2}}{(c+1)_{n-2}(n-2)!}\frac{(c)_n\,n!}{(-n)_n(b)_n} = \frac{(n-1)(c+n-1)(b-c)}{(b+n-2)(b+n-1)^2}$$

and, for $k \in \{2, \ldots, n-1\}$,

$$c_k = \frac{\theta_{k+1}}{\theta_{k-1}}\frac{(-n+k+1)_{n-k-1}(b+1-k)_{n-k-1}}{(c+1)_{n-k-1}(n-k-1)!}\frac{(n-k+1)!(c+1)_{n-k+1}}{(-n+k-1)_{n-k+1}(b+3-k)_{n-k+1}}$$
$$= \frac{(n-k)(n-k+c)(b-k+1)(-c+1+b-k)}{(n-2k+b+2)(n-2k+b)(b-2k+1+n)^2}.$$

From Lemma 10, we know that $c_k > 0$ for all $k \in \{1, \ldots, n-1\}$ when $(c, b) \in \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4$. The result now follows from Theorem 3. ∎

Proof of Theorem 6. From [2], Corollary 4 (i), we know that for $-1 < \alpha < 0$ and $-1 < \beta < 0$, the Jacobi polynomials $P_n^{(\alpha-1, \beta-1)}(x)$ have real simple zeros and $(n-2)$ of them are in the interval $(-1, 1)$. The smallest zero is smaller than -1 and the largest zero is larger than 1. Equivalently, the same is true for the zeros of Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ when $-2 < \alpha < -1$ and $-2 < \beta < -1$.

(i) One of the connections between Jacobi polynomials and the polynomials ${}_2F_1(-n, b; c; z)$ is given by (cf. [25], p. 254, eq. 3)

$$P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n(1+\beta)_n}{n!}\,{}_2F_1\left(-n, 1+\alpha+\beta+n; 1+\beta; \frac{x+1}{2}\right) \tag{6}$$

where $\alpha = b - n - c$ and $\beta = c - 1$. The conditions $-2 < \beta < -1$ and $-2 < \alpha < -1$ are equivalent to $-1 < c < 0$ and $c + n - 2 < b < c + n - 1$. Furthermore, the intervals $(-1, 1)$, $(1, \infty)$ and $(-\infty, -1)$ are transformed to $(0, 1)$, $(1, \infty)$ and $(-\infty, -1)$ respectively under the linear mapping $x = 2z - 1$. Thus, when $c \in (-1, 0)$ and $b \in (c+n-2, c+n-1)$, ${}_2F_1(-n, b; c; z)$ has $n - 2$ real, simple zeros in the interval $(0, 1)$, one zero in $(1, \infty)$ and one zero in $(-\infty, 0)$ for each $n \in \mathbb{N}$.

(ii) The representation ([25], p. 255, eq. 9)

$$P_n^{(\alpha,\beta)}(x) = \frac{(1 + \alpha + \beta)_{2n}}{n!(1 + \alpha + \beta)_n} \left(\frac{x+1}{2}\right)^n {}_2F_1\left(-n, -\beta - n; -\alpha - \beta - 2n; \frac{2}{x+1}\right) \tag{7}$$

where $\alpha = b - c - n$ and $\beta = -b - n$ yields the stated result, since the restrictions $-2 < \alpha < -1$ and $-2 < \beta < -1$ will correspond to $1 + b - n < c < 2 + b - n$ and $1 - n < b < 2 - n$ while the intervals $-1 < x < 1$, $x > 1$ and $x < -1$ are mapped to $z > 1$, $0 < z < 1$ and $z < 0$ respectively under the fractional transformation $z = \frac{2}{x+1}$.

(iii) For this case we use the representation (cf. [25], p. 255, eq. 8)

$$P_n^{(\alpha,\beta)}(x) = \frac{(1 + \beta)_n}{n!} \left(\frac{x-1}{2}\right)^n {}_2F_1\left(-n, -\alpha - n; \beta + 1; \frac{x+1}{x-1}\right) \tag{8}$$

where $\alpha = -b - n$ and $\beta = c - 1$. Under the transformation $z = \frac{x+1}{x-1}$, the interval $-1 < x < 1$ is mapped to the negative real line while the intervals $x > 1$ and $x < -1$ are mapped to $z = 1 + \frac{2}{x-1} > 1$ and $0 < z < 1$ respectively. Also, since $-2 < \alpha < -1$ and $-2 < \beta < -1$ correspond to $1 - n < b < 2 - n$ and $-1 < c < 0$, the result follows.

■

Proof of Theorem 7. From [2], Corollary 4 (ii) (a), we know that for $-1 < \alpha$ and $-1 < \beta < 0$, the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta-1)}(x)$ are all real and simple, and $(n-1)$ of them lie in the interval $(-1, 1)$. The smallest zero is less than -1. We may equivalently say the same about the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ for $\alpha > -1$ and $-2 < \beta < -1$. Using transformations (6), (7) and (8) the results can be proven following the same reasoning as in the proof of Theorem 6.

■

Proof of Theorem 8. From [2], Corollary 4 (ii) (b), we know that for $-1 < \alpha < 0$ and $-1 < \beta$, the zeros of the Jacobi polynomial $P_n^{(\alpha-1,\beta)}(x)$ are all real and simple, and $(n-1)$ of them lie in the interval $(-1, 1)$. Moreover, the largest zero is greater than 1. We may equivalently say the same about the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ for $-2 < \alpha < -1$ and $\beta > -1$. The results now follow from (6), (7) and (8) as before.

■

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